Enhancements for Reduced Basis Methods: Reducing Offline Computational Costs

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EAS 621 Fall 2013
UMass Dartmouth

April 24, 2014

Outline

- Background
- Reduced Basis Method (RBM)
- Greedy Algorithm
- Goals

Parametric Partial Differential Equations

- Parametric partial differential equations is a large field in scientific computing
- These are problems where certain coefficients stay constant throughout the simulation
- Those coefficients may represent physical properties of the problem

Examples

- Transport equation
  - Wave number μ
  - Source term dependent on the parameter values

$$U_t + \mu U_x = F_\mu$$

- Heat equation
  - With different diffusion coefficients μ_1 and μ_2
  - Solution U is dependent on the t, X, Y and fixed μ_1 and μ_2.

$$U_t = \mu_1 U_{xx} + \mu_2 U_{yy}$$

Parametric partial differential equations (μPDE) may be represented as:

$$L_\mu u_\mu = f_\mu$$

Example:

$$(1 + \mu_1 x) u_{xx} + (1 + \mu_2 y) u_{yy} = f(x, \mu)$$

$$(1 + \mu_1 x) D_{xx} u + (1 + \mu_2 y) D_{yy} u = f(x, \mu)$$

$$L_\mu u_\mu = f_\mu$$

Problems

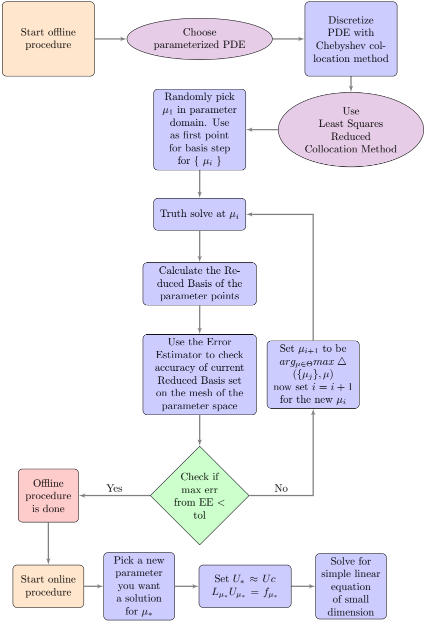
- Calculating the exact solution is costly
- Every time the parameters are changed the exact solution has to be recomputed
- Many problems require a large amount of solutions calculated very quickly with a reasonable degree of accuracy

So how can RBM solve these problems?

RBM

- Solving a μPDE is separated into two parts:
  1. Offline: precompute multiple solutions for key parameter values
  2. Online: uses precomputed solutions for simple linear combination
- Extra work in offline portion allows very large speed increase for on-the-fly computations

Original RBM



Offline

- Choose first μ_1 randomly
- Compute $u^N_{\mu_1}$ the true solution using least squares (expensive)

$$A^T Ac = A^T f$$

$$A = Lu_{\mu_1}$$

- Construct $u^{RB}_{\mu} = c_1 u^N_{\mu_1}$
- Choose μ_2 by looking at the residual and pick the μ that maximizes
- For each μ construct $u^{RB}_{\mu} = c_1 u^N_{\mu_1} + c_2 u^N_{\mu_2}$

Online

Using the set of precomputed solutions $\{u_{\mu_i}^N\}$ approximate

$$L_{\mu^\star} u_{\mu^\star} \approx f(x, \mu^\star)$$

Combine $\{u_{\mu_i}^N\}$ to estimate the solution of the PDE for $\mu^\star$

$$u_{\mu^\star} = \Sigma c_i u_{\mu_i}^N$$

$$L_{\mu^\star} \Sigma c_i u_{\mu_i}^N = f(x, \mu^\star)$$

$$\Sigma c_i [L_{\mu^\star} u_{\mu_i}^N] = f(x, \mu^\star)$$

Current Parameter Selection Procedure

1). Form $\mathbb{A}_{i-1} = (\mathbb{L}_{\mathcal{N}} u^{\mathcal{N}}_{\mu^1}, \mathbb{L}_{\mathcal{N}} u^{\mathcal{N}}_{\mu^2}, \dots, \mathbb{L}_{\mathcal{N}} u^{\mathcal{N}}_{\mu^{i-1}})$.

2). For all $\mu \in \Xi$, solve $\mathbb{A}^T_{i-1}\mathbb{A}_{i-1}\vec{c} = \mathbb{A}^T_{i-1} f^{\mathcal{N}}$ to obtain $u^{(i-1)}_{\mu} = \sum_{j=1}^{1-1} c_j u^{\mathcal{N}}_{\mu^j}$.

3). For all $\mu \in \Xi$, calculate $\Delta_{i-1}(\mu)$

4). Set $\mu^i = argmax_{\mu} \Delta_{i-1}(\mu)$.

5). Solve $\mathbb{L}_{\mathcal{N}}(\mu^i) u^{\mathcal{N}}_{\mu^i}(x) = f(x; \mu^i)$ for $x \in C^{\mathcal{N}}$.

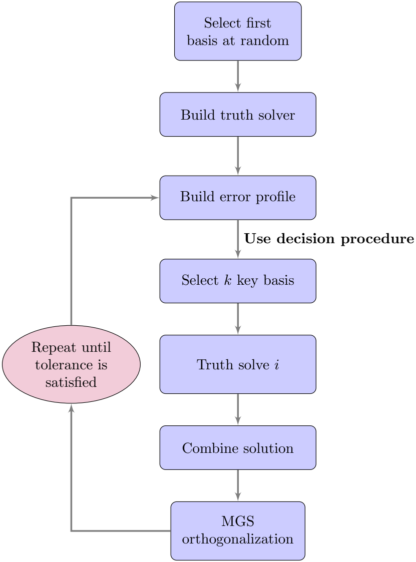
Building the Error Profile

- Error Estimator computes the modified residual $\Delta_{i-1}(\mu)$

$$\frac{|f_\mu - L_\mu u_\mu^{RB}|}{\lambda_\mu(L_\mu^T L_\mu)}$$

- Pick the μ that maximizes the residual

Greedy Algorithm



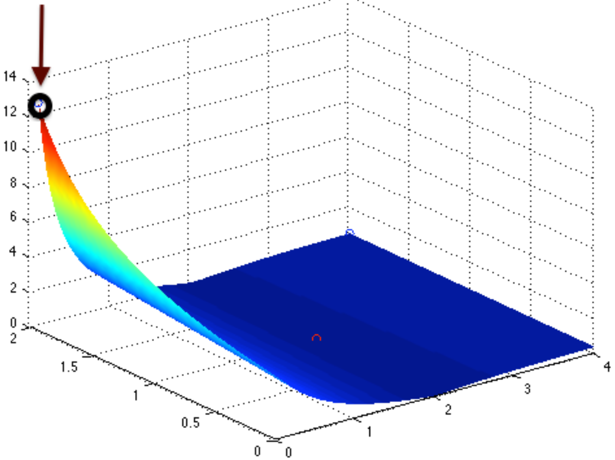
Test Case One: AnisoWave

Anisotropic wave speed simulation problem:

$$-u_{xx} - \mu_1 u_{yy} - \mu_2 u = f$$

$$\text{on}[-1,1] \times [-1,1] \text{ with } u = 0 \text{ on boundary}$$

Error Profile After First Iteration



Examples of Error Profile Using Greedy Algorithm

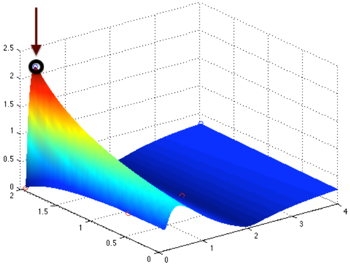


Figure: Second iteration

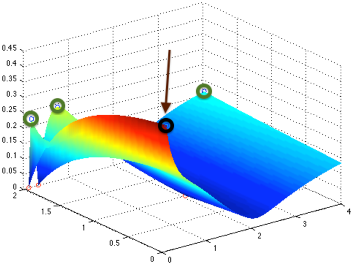
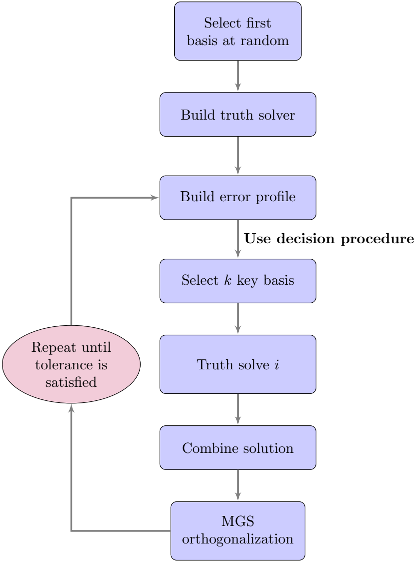


Figure: Third iteration

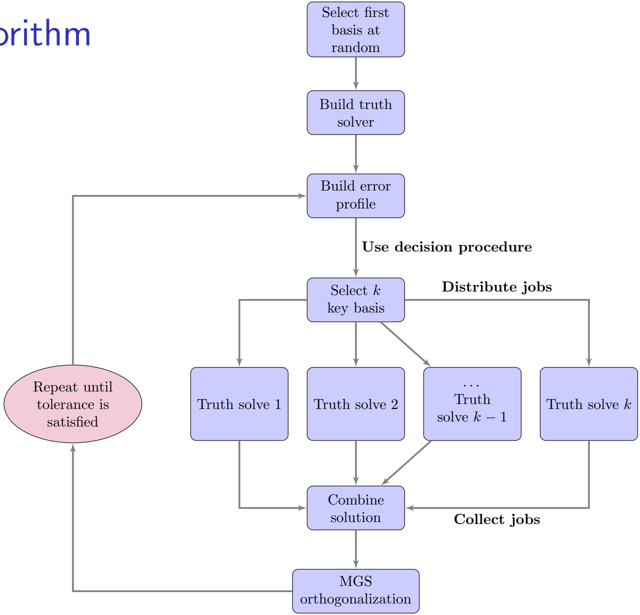
Goals

1. Reduce the computational costs of the offline procedure.
2. Create an Interactive Parameter Selection
3. Automate the selection of multiple parameters
4. Parallel processing of RBM algorithm

Greedy Algorithm



Greedy Algorithm



Thank You

Questions?

Works Cited

Yanlai Chen and Sigal Gottlieb. *Reduced Collocation Methods: Reduced Basis Methods in the Collocation Framework*. arXiv1201.4188v2[math.NA], 2012.